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Proof of Uniqueness for Gross-Pitaevskii Hierarchy via Board Games

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RTG - ANALYSIS of PDE
DEPARTMENT OF MATHEMATICS

Bose-Einstein Condensates

When a dilute gas of identical bosons is cooled to a low temperature, the system stops acting as many individual particles and instead behaves in a way we would expect a single particle to act. First predicted in the 1920s by Albert Einstein and Satyendra Nath Bose, this 'fifth state of matter' was not observed in a lab until 1995 with the works of Cornell-Wieman and Ketterle.

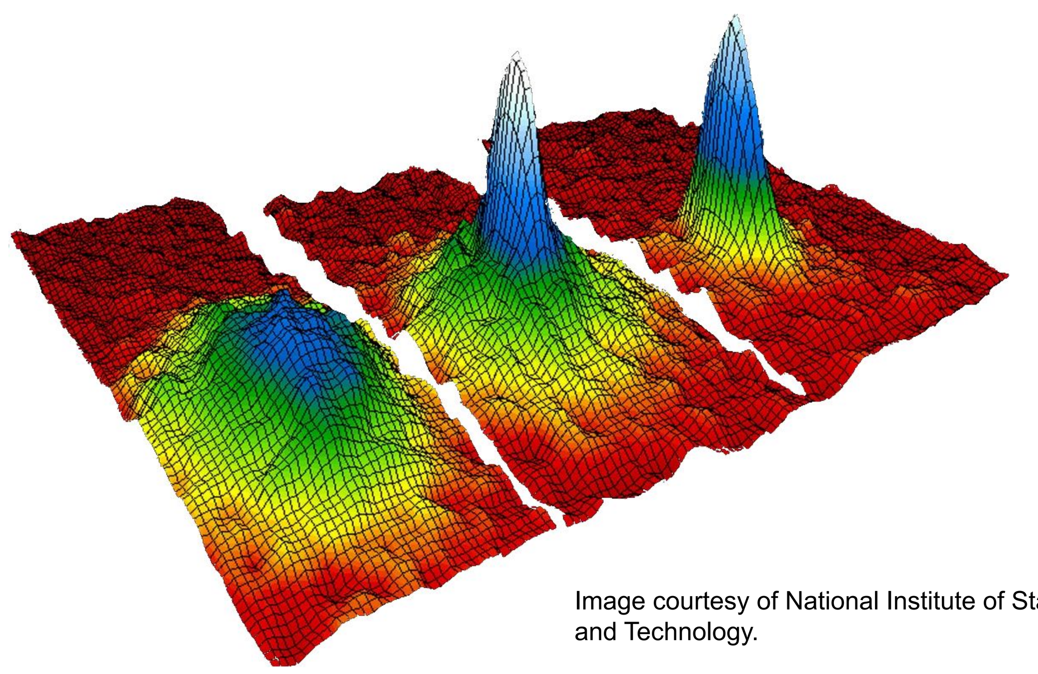


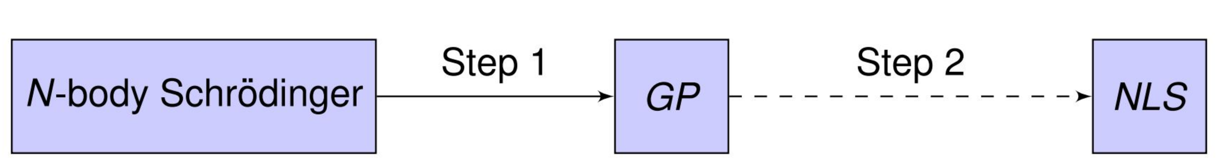
Image courtesy of National Institute of Standards and Technology.

Here is the measured velocity distribution of a gas of rubidium atoms before (left) and after (right) the appearance of a Bose-Einstein Condensate. Notice that once the gas has been cooled, the system becomes more predictable in the sense that the wave function condenses as the wave functions for individual particles become nearly identical.

Modeling BECs

In a first approach to modeling a large system of particles, one might consider using the N-particle linear Schrödinger equation. However, in practice these gases have a number of particles many of orders of magnitude higher than a reasonable amount to consider.

However, taking a limit as the number of particles goes to infinity offers hope of a more reasonable model. If we can show that solutions of this problem converge to solutions of the Gross-Pitaevskii hierarchy of equations (step 1) all that is left is showing uniqueness of these solutions to non-linear Schrödinger (step 2).



N-Body Schrödinger

The N-Body Schrödinger equation is a means of describing the time-evolution of N identical quantum particles.

$$i\partial_t \Phi_N = H_N \Phi_N$$

$$H_N = \sum_{j=1}^{N}(-\Delta_{x_j}) + \frac{1}{N}\sum_{1\le i<j\le N} V_N(x_i - x_j)$$

Each Φ_N is a symmetric $L^2(\mathbb{R}^{3N})$ function. By considering finite marginals and a limit as N goes to infinity, we converge to the Gross-Pitaevskii hierarchy of equations. A detailed proof of this convergence is given in 'Derivation of the Cubic Non-linear Schrödinger equation from Quantum dynamics of Many-Body Systems' by Erdos, Schlein, and Yau.

Gross-Pitaevskii Hierarchy

The Gross-Pitaevskii hierarchy is a set of equations which describe the state of a quantum system with a large number of identical particles. A solution to the hierarchy is a sequence of functions $\gamma^{(k)} \in L^2(\mathbb{R} \times \mathbb{R}^{3k} \times \mathbb{R}^{3k})$ which satisfy the following:

- $\gamma^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k) = \overline{\gamma^{(k)}(t, \mathbf{x}'_k, \mathbf{x}_k)}$
- $\gamma^{(k)}(t, x_{\sigma(1)}, \cdots x_{\sigma(k)}, x'_{\sigma(1)}, \cdots x'_{\sigma(k)}) = \gamma^{(k)}(t, x_1, \cdots x_k, x'_1, \cdots x'_k)$
- $(i\partial_t + \Delta_{\mathbf{x}_k} - \Delta_{\mathbf{x}'_k})\,\gamma^{(k)} = \sum_{j=1}^{k} B_{j,k+1}(\gamma^{(k+1)})$
- Initial conditions: $\gamma^{(k)}(0, \mathbf{x}_k, \mathbf{x}'_k) = \gamma_0^{(k)}(\mathbf{x}_k, \mathbf{x}'_k)$

The first two conditions say that the functions should be symmetric and independent of arrangement of spatial inputs. For the third, the evaluation operator $B_{j,k}$ is introduced to explain that the time-evolution of a system of k+1 particles should depend on how the k-particle system would evolve.

The hierarchy is particularly nice because we can verify that factors of the solution solve 3+1-dimensional non-linear Schrödinger:

$$\gamma^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k) = \prod_{j=1}^{k} \phi(t, x_j)\overline{\phi}(t, x'_j)$$

where each ϕ solves $(i\partial_t + \Delta)\,\phi = \phi|\phi|^2, \quad \phi(0,x) = \phi(x)$

Uniqueness of Solutions

Both convergence of N-body Schrödinger to the GP hierarchy and uniqueness of solutions were shown by Erdos, Schlein, and Yau with a rather complicated argument of counting Feynman diagrams. The paper 'On the Uniqueness of Solutions to the Gross-Pitaevskii Hierarchy' by Klainerman and Machedon aims to simplify the uniqueness proof by constructing a more easily visualized combinatorial argument with an estimate in a different space.

Theorem 1.1 (Main Theorem). *Consider solutions $\gamma^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k)$ of the Gross-Pitaevskii hierarchy (2), with zero initial conditions, which verify the estimates,*

$$\int_0^T \|R^{(k)} B_{j,k}\gamma^{(k)}(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^{3k}\times\mathbb{R}^{3k})}\, dt \le C^k \qquad (8)$$

for some $C > 0$ and all $1 \le j < k$. Then $\|R^{(k)}\gamma^{(k)}(t,\cdot,\cdot)\|_{L^2(\mathbb{R}^{3k}\times\mathbb{R}^{3k})} = 0$ for all k and all t.

The interesting part of this proof comes from using Duhamel expansion of $\gamma^{(k)}$ to rewrite it as an integral across the sum of all possible histories of the system. In contrast to Erdos, Schlein, and Yau, the Klainerman-Machedon paper organizes terms via a board game argument as opposed to Feynman diagrams. Naively applying triangle inequality to get an estimate for our integral does not work because the number of possible histories grows factorially with n.

$$\begin{pmatrix} t_2 & t_5 & t_4 & t_3 \\ \mathbf{B_{1,2}} & B_{1,3} & \mathbf{B_{1,4}} & B_{1,5} \\ 0 & \mathbf{B_{2,3}} & B_{2,4} & B_{2,5} \\ 0 & 0 & B_{3,4} & B_{3,5} \\ 0 & 0 & 0 & \mathbf{B_{4,5}} \end{pmatrix} \qquad \begin{pmatrix} t_2 & t_4 & t_5 & t_3 \\ \mathbf{B_{1,2}} & \mathbf{B_{1,3}} & B_{1,4} & B_{1,5} \\ 0 & B_{2,3} & \mathbf{B_{2,4}} & B_{2,5} \\ 0 & 0 & B_{3,4} & \mathbf{B_{3,5}} \\ 0 & 0 & 0 & B_{4,5} \end{pmatrix}$$

One possible history of a 5-particle system and an acceptable move on the board.

The matrix on the left can be thought of as a board for some game where moves (changes to the matrix) preserve the integral across all histories. Showing that there are; reduced forms of all possible boards, a finite number of moves to reduce all boards, and then at most 4^n reduced boards with a series of lemmas, the proof of the main theorem becomes a simple sequence of applying Holders and an estimate on the homogeneous problem n-1 times.

The board game argument has since been improved upon in 'Unconditional Uniqueness for the Cubic Gross-Pitaevskii Hierarchy via Quantum de Finetti' by Chen, Hainzl, Pavlović, and Seiringer to get unconditional uniqueness of solutions. The space is that of the original Erdos, Schlein, and Yau paper.

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Klainerman, S., Machedon, M. On the Uniqueness of Solutions to the Gross-Pitaevskii Hierarchy. *Commun. Math. Phys.* 279, 169–185 (2008). https://doi.org/10.1007/s00220-008-0426-4